Identități trigonometrice

Metoda 1.

Considerăm sumele:

$$S_1=\cos\frac{\pi}{n}+\cos\frac{2\pi}{n}+\cos\frac{3\pi}{n}+\ldots+\cos\frac{(n-1)\pi}{n};$$

$$S_2=\sin\frac{\pi}{n}+\sin\frac{2\pi}{n}+\sin\frac{3\pi}{n}+\ldots+\sin\frac{(n-1)\pi}{n}.$$

Pentru a calcula cele două sume, vom considera S_1+iS_2 în locul sumelor separate.

$$S_1+iS_2=\left(\cos\frac{\pi}{n}+\cos\frac{2\pi}{n}+\cos\frac{3\pi}{n}+\ldots+\cos\frac{(n-1)\pi}{n}\right)+$$

$$+i\left(\sin\frac{\pi}{n}+\sin\frac{2\pi}{n}+\sin\frac{3\pi}{n}+\ldots+\sin\frac{(n-1)\pi}{n}\right)=\left(\cos\frac{\pi}{n}+i\sin\frac{\pi}{n}\right)+$$

$$+\left(\cos\frac{2\pi}{n}+i\sin\frac{2\pi}{n}\right)+\left(\cos\frac{3\pi}{n}+i\sin\frac{3\pi}{n}\right)+\ldots+\left(\cos\frac{(n-1)\pi}{n}+i\sin\frac{(n-1)\pi}{n}\right)=$$

$$=\left(\cos\frac{\pi}{n}+i\sin\frac{\pi}{n}\right)+\left(\cos\frac{\pi}{n}+i\sin\frac{\pi}{n}\right)^2+\left(\cos\frac{\pi}{n}+i\sin\frac{\pi}{n}\right)^3+$$

$$+\ldots+\left(\cos\frac{\pi}{n}+i\sin\frac{\pi}{n}\right)^{n-1}.$$

Am obținut o sumă de termeni în progresie geometrică cu rația $\boxed{q=\cos\frac{\pi}{n}+i\sin\frac{\pi}{n}}$

Prin urmare,

$$S_1 + iS_2 = \frac{\left(\cos\frac{\pi}{n} + i\sin\frac{\pi}{n}\right)\left[\left(\cos\frac{\pi}{n} + i\sin\frac{\pi}{n}\right)^{n-1} - 1\right]}{\cos\frac{\pi}{n} + i\sin\frac{\pi}{n} - 1} =$$

$$= \frac{\left(\cos\frac{\pi}{n} + i\sin\frac{\pi}{n}\right)^{n} - \left(\cos\frac{\pi}{n} + i\sin\frac{\pi}{n}\right)}{\cos\frac{\pi}{n} + i\sin\frac{\pi}{n} - 1} =$$

$$= \frac{\cos\frac{n\pi}{n} + i\sin\frac{n\pi}{n} - \cos\frac{\pi}{n} - i\sin\frac{\pi}{n}}{\cos\frac{\pi}{n} + i\sin\frac{\pi}{n} - 1} =$$

$$= \frac{-1 - \cos\frac{\pi}{n} - i\sin\frac{\pi}{n}}{\cos\frac{\pi}{n} + i\sin\frac{\pi}{n} - 1} = \frac{2\cos^2\frac{\pi}{2n} + 2\sin\frac{\pi}{2n}\cos\frac{\pi}{2n}i}{2\sin^2\frac{\pi}{2n} - 2\sin\frac{\pi}{2n}\cos\frac{\pi}{2n}i} =$$

$$= \frac{2\cos\frac{\pi}{2n}\left(\cos\frac{\pi}{2n} + i\sin\frac{\pi}{2n}\right)}{2\sin\frac{\pi}{2n}\left(\sin\frac{\pi}{2n} - i\cos\frac{\pi}{2n}\right)} = ctg\frac{\pi}{2n}\,\frac{i\left(\sin\frac{\pi}{2n} - i\cos\frac{\pi}{2n}\right)}{\left(\sin\frac{\pi}{2n} - i\cos\frac{\pi}{2n}\right)},$$

$$= \boxed{0 + i\cdot ctg\frac{\pi}{2n}}$$

Din egalitatea $S_1 + iS_2 = 0 + i\cdot ctg\frac{\pi}{2n}$, egalând părțile reale şi cele imaginare ale celor două numere complexe, avem:

$$\boxed{\begin{aligned} &S_1 = 0 \\ &S_2 = ctg\frac{\pi}{2n} \end{aligned}}$$

Metoda 2.

Vom folosi identitățile trigonometrice:

$$\boxed{\begin{aligned} &\sin b\cos a = \frac{1}{2}\left[\sin(a+b) - \sin(a-b)\right],\ \forall a,b\in\mathbb{R} \\ &\sin a\sin b = \frac{1}{2}\left[\cos(a-b) - \cos(a+b)\right],\ \forall a,b\in\mathbb{R} \end{aligned}}$$

Vom înmulți S_1 şi S_2 cu $\sin\frac{\pi}{2n}$.

Avem egalitățile următoare (pentru S_1) pe care le vom aduna :

$$\sin\frac{\pi}{2n}\cos\frac{\pi}{n}=\frac{1}{2}\left[\sin\frac{3\pi}{2n}-\sin\frac{\pi}{2n}\right]$$

$$\sin\frac{\pi}{2n}\cos\frac{2\pi}{n}=\frac{1}{2}\left[\sin\frac{5\pi}{2n}-\sin\frac{3\pi}{2n}\right]$$

$$\sin\frac{\pi}{2n}\cos\frac{3\pi}{n}=\frac{1}{2}\left[\sin\frac{7\pi}{2n}-\sin\frac{5\pi}{2n}\right]$$

$$\vdots \qquad\qquad \vdots$$

$$\sin\frac{\pi}{2n}\cos\frac{(n-2)\pi}{n}=\frac{1}{2}\left[\sin\frac{(2n-3)\pi}{2n}-\sin\frac{(2n-5)\pi}{2n}\right]$$

$$\sin\frac{\pi}{2n}\cos\frac{(n-1)\pi}{n}=\frac{1}{2}\left[\sin\frac{(2n-1)\pi}{2n}-\sin\frac{(2n-3)\pi}{2n}\right]$$

$$\sin\frac{\pi}{2n}S_1=\frac{1}{2}\left[\sin\frac{(2n-1)\pi}{2n}\right]-\sin\frac{\pi}{2n}=\frac{1}{2}\left[\sin\left(\pi-\frac{\pi}{2n}\right)-\sin\frac{\pi}{2n}\right]=\frac{1}{2}\left(\sin\frac{\pi}{2n}-\sin\frac{\pi}{2n}\right)$$

adică,

$$\sin\frac{\pi}{2n}S_1=0\Rightarrow \boxed{S_1=0}$$

Analog, pentru $\sin\frac{\pi}{2n}S_2$ vom avea după reduceri:

$$\sin\frac{\pi}{2n}S_2=\cos\frac{\pi}{2n}\Rightarrow \boxed{S_2=\frac{\cos\frac{\pi}{2n}}{\sin\frac{\pi}{2n}}=ctg\frac{\pi}{2n}}$$