3241. [2007 : 237, 239] *Proposed by Virgil Nicula, Bucharest, Romania.*

Let a, b, c be any real numbers such that $a^2+b^2+c^2=9$. Prove that

$$3\cdot\min\{a,b,c\} \le 1+abc.$$

Solution by Arkady Alt, San Jose, CA, USA, modified by the editor.

Without loss of generality, we may assume $a \le b \le c$. We want to prove the inequality $abc+1 \ge 3a$.

If $a \le 0$, then using the inequality $bc \le \frac{1}{2}(b^2+c^2)$, we obtain

$$\begin{aligned} abc+1-3a &\ge a|bc|-3a+1 \ge \tfrac{1}{2}a(b^2+c^2)-3a+1 \\ &= \tfrac{1}{2}a(9-a^2)-3a+1 = \tfrac{1}{2}(a+1)^2(2-a) \ge 0, \end{aligned}$$

with equality if and only if $a=-1$ and $b=c=2$.

Now, let $a>0$. Since $a \le b \le c$, we get $9=a^2+b^2+c^2 \ge 3a^2$; whence, $a \le \sqrt{3}$. We have the obvious inequality $(c^2-a^2)(b^2-a^2) \ge 0$, which yields

$$bc \ge a\sqrt{c^2+b^2-a^2} = a\sqrt{9-2a^2}.$$

Hence, it suffices to prove the (stronger) inequality

$$a\left(a\sqrt{9-2a^2}\right)+1 \ge 3a.$$

If $0<a\le 1$, then $\sqrt{9-2a^2} \ge \sqrt{7} > \frac{9}{4}$; thus, $4a^2\sqrt{9-2a^2} > 9a^2$. Using the inequality $(t+1)^2 \ge 4t$, we obtain

$$\left(a^2\sqrt{9-2a^2}+1\right)^2 \ge 4a^2\sqrt{9-2a^2} > 9a^2,$$

which yields

$$a\left(a\sqrt{9-2a^2}\right)+1 \ge 3a.$$

If $1<a\le\sqrt{3}$, then we can prove an equivalent form of our inequality, namely, the one obtained by squaring both sides of

$$a\left(a\sqrt{9-2a^2}\right) \ge 3a-1;$$

that is,

$$2a^6-9a^4+9a^2-6a+1 \le 0.$$

We have

$$\begin{aligned} 2a^6-9a^4+9a^2-6a+1 &< 2a^6-9a^4+9a^2-5 \\ &= (2a^2-1)(a^2-1)(a^2-3)-(a^2+2) \\ &< 0, \end{aligned}$$

which completes the proof.

Also solved by MOHAMMED AASSILA, Strasbourg, France; ROY BARBARA, Lebanese University, Fanar, Lebanon; JOHN HAWKINS and DAVID R. STONE, Georgia Southern University, Statesboro, GA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong, China; SALEM MALIKIĆ, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; ALEX REMOROV, student, William Lyon Mackenzie Collegiate Institute, Toronto, ON; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.