Sumă de puteri

Fie ecuația

$$z+\frac{1}{z}=-2\cos x,\ \ x\in\mathbb{R}$$

Calculați $z^{n}+\frac{1}{z^{n}}$, $n\in\mathbb{N}$.

Rezolvăm ecuația.

$$z+\frac{1}{z}=-2\cos x \Leftrightarrow z^{2}+2z\cos x+1=0 \qquad (1)$$
$$\Delta=4\cos^{2}x-4=4\left(\cos^{2}x-1\right)=-4\sin^{2}x$$

Rădăcinile pătrate ale lui Δ sunt $\pm 2i\sin x$.

Soluțiile ecuației (1) sunt:

$z_{1,2}=-\cos x\pm i\sin x=\cos(\pi-x)\pm i\sin(\pi-x)$

Cazul I $z_{1}=\cos(\pi-x)+i\sin(\pi-x)$

$$E=z_{1}^{n}+\frac{1}{z_{1}^{n}}=\left[\cos(\pi-x)+i\sin(\pi-x)\right]^{n}+\frac{1}{\left[\cos(\pi-x)+i\sin(\pi-x)\right]^{n}}=$$
$$\overset{Moivre}{=}\cos n(\pi-x)+i\sin n(\pi-x)+\frac{1}{\cos n(\pi-x)+i\sin n(\pi-x)}=$$
$$=\cos n(\pi-x)+i\sin n(\pi-x)+\frac{\cos n(\pi-x)-i\sin n(\pi-x)}{\left(\cos n(\pi-x)+i\sin n(\pi-x)\right)\left(\cos n(\pi-x)-i\sin n(\pi-x)\right)}=$$
$$=\cos n(\pi-x)+i\sin n(\pi-x)+\frac{\cos n(\pi-x)-i\sin n(\pi-x)}{\underbrace{\cos^{2}n(\pi-x)+\sin^{2}n(\pi-x)}_{=1}}=$$
$$=\cos n(\pi-x)+i\sin n(\pi-x)+\cos n(\pi-x)-i\sin n(\pi-x)=2\cos n(\pi-x)$$

Cazul II $z_{2}=\cos(\pi-x)-i\sin(\pi-x)$

$$E = z_2^n + \frac{1}{z_2^n} = \left[\cos(\pi - x) - i\sin(\pi - x)\right]^n + \frac{1}{\left[\cos(\pi - x) - i\sin(\pi - x)\right]^n} =$$

$$\overset{Moivre}{=} \cos n(\pi - x) - i\sin n(\pi - x) + \frac{1}{\cos n(\pi - x) - i\sin n(\pi - x)} =$$

$$= \cos n(\pi - x) - i\sin n(\pi - x) + \frac{\cos n(\pi - x) + i\sin n(\pi - x)}{\left(\cos n(\pi - x) - i\sin n(\pi - x)\right)\left(\cos n(\pi - x) + i\sin n(\pi - x)\right)} =$$

$$= \cos n(\pi - x) - i\sin n(\pi - x) + \frac{\cos n(\pi - x) + i\sin n(\pi - x)}{\underbrace{\cos^2 n(\pi - x) + \sin^2 n(\pi - x)}_{=1}} =$$

$$= \cos n(\pi - x) - i\sin n(\pi - x) + \cos n(\pi - x) + i\sin n(\pi - x) = 2\cos n(\pi - x)$$

Observații.

1. Am utilizat formula lui *Moivre*:

"*Oricare ar fi* $t \in \mathbb{R}$ *şi oricare ar fi* $n \in \mathbb{N}$, *avem* $\boxed{(\cos t + i\sin t)^n = \cos nt + i\sin nt}$"

2. Dacă $n \in \mathbb{N}$, atunci:

$$\boxed{(\cos t + i\sin t)^{-n} = \frac{1}{\cos nt + i\sin nt} = \cos(-nt) + i\sin(-nt)}$$

3. Dacă $n \in \mathbb{N}$, atunci $z^{-n} + \frac{1}{z^{-n}} = \frac{1}{z^n} + z^n = z^n + \frac{1}{z^n}$.

4. Rezultatul mai poate fi discutat în funcție de paritatea lui n.